1 Basic Terminology

2 Connectivity

Def. A **connected graph** G is a graph where $\forall u, v \in V(G), \exists$ walk from $u \to v$

Def. $X_1, ..., X_n$ is a **partition** of $V(G)$ if $X_1 \cup ... \cup X_n = V(G)$ and $X_1 \cap ... \cap X_n = \emptyset$

Lemma 2.1. If G is not a connected graph then there exists a partition (X, Y) of $V(G)$ s.t. $X, Y \neq \emptyset$ and no edge links X and Y

Lemma 2.2. If G is connected then $\forall$ partition (X, Y) of $V(G)$ s.t. $X, Y \neq \emptyset : \exists e \in E(G)$ s.t. e has an end in both X and Y.

Lemma 2.3. If $\exists$ walk $u \to v$ then $\exists$ path $u \to v$

Lemma 2.4. H_1, H_2 connected subgraphs s.t. $V(H_1) \cap V(H_2) \neq \emptyset \implies H_1 \cup H_2$ connected

Def. A **component** of a graph G, denoted is a maximal connected subgraph. We say a subgraph $H \in G$ is **maximal** if $\forall H'$ s.t. $H \subseteq H' \subseteq G$ we have $H' = H$. Denote the number of distinct connected components by $comp(G)$

Lemma 2.5. Every vertex of a graph G belongs to a unique connected component of G.

Lemma 2.6. If $e \in E(G)$ then either:

1. e belongs to a cycle in G and $comp(G - e) = comp(G)$
2. e belongs to no cycle and $comp(G - e) = comp(G) + 1$

3 Trees and Forests

Def. A **forest** is a graph with no cycles

Def. A **tree** is a connected forest

Def. A **leaf** is a vertex of degree 1.

Theorem 3.1. If F is a non-empty forest then $comp(F) = |V(F)| - |E(G)|$

Corollary 3.2. If T is a tree then $|E(T)| = |V(T)| - 1$

Def. A map $f : V(G) \cup E(G) \to V(H) \cup E(H)$ is an isomorphism between G and H if f is a bijection from $V(G) \to V(H)$ and $E(G) \to E(H)$ and $v \in V(G)$ incident $e \in E(G) \iff f(v) \in V(H)$ incident to $f(e) \in E(H)$

Lemma 3.3. T is a tree s.t. $|V(T)| \geq 2$, X a set of leaves and Y is a set of vertices with degree larger than 3 $\implies |X| \geq |Y| + 2$ (and in particular $|X| \geq 2$)

Lemma 3.4. T tree s.t. $|V(T)| \geq 2$. T has exactly two leaves then T is a path.

Lemma 3.5. G graph, v leaf of G then G is a tree if an only if $G \setminus v$ is a tree.

Lemma 3.6. Let T be a tree then for any $a, b \in V(T)$ there exists a unique path $P \subseteq T$ from a to b.

4 Spanning Trees

Def. A **spanning tree** is a tree $T \subseteq G$ with $V(T) = V(G)$.

- G has a spanning tree iff it is connected and non-null.

Lemma 4.1. Let G be non-null and connected. If $H \subseteq G$ minimal such that $V(H) = V(G)$ and H connected $\implies$ H is a spanning tree.

Lemma 4.2. Let G be non-null and connected. If $H \subseteq G$ is a maximal subgraph such that G has no cycles, it is a spanning tree.

Def. Let G be a graph, T a spanning tree of G. For $f \in E(G) - E(T)$. Let C be a cycle of G such that $C - f$ is a path in T. Then we call C the **fundamental cycle** of f with respect to T.

Lemma 4.3. Let T be a spanning tree of G. Let $f \in E(G) - E(T)$. Then, there exists a unique fundamental cycle of f with respect to T.

Lemma 4.4. Let T be a spanning tree of G. Let $f \in E(G) - E(T)$ and C be the fundamental cycle of f with $e \in E(C) - \{f\}$. Then $T + \{f\} - \{e\}$ is still a spanning tree of G

Def. Let G be a graph, $w : E(G) \to \mathbb{R}^+$. A spanning tree T is called a **min-cost spanning tree**(MST) of G if:

$$\sum_{e \in E(G)} w(e)$$

is minimal among all spanning trees.

Corollary 4.5. Let T be a MST for a graph G with weight function w. Let f, c, e be as in the statement of 4.4. Then, $w(f) \geq w(e)$.

Theorem 4.6. Let T be a MST on n vertices and assume for convenience that $w(e)$ is distinct for all edges $e \in E(G)$. Let $e_1, ...e_{n-1}$ be the edges of T with $w(e_1) < w(e_2)$....

Then e_i is the edge with minimum weight such that $e_i \not\in \{e_1, e_2...e_{i-1}\}$ and such that $\{e_1, e_2...e_i\}$ contains no cycle.

Def. Kruskal's Algorithm: Given a graph G, outputs a MST. With the first $i-1$ edges chosen, pick the edge with minimum weight such that adding the edge doesn't create a cycle.

Theorem 4.7. Kruska's algorithm always outputs a MST.

Def. A **rooted forest** F is a forest with a vertex (called **root**) selected in every component.

Theorem 4.8. Cayley's theorem: The complete graph on n vertice has n^{n-2} spanning trees.

Theorem 4.9. There are $\binom{n}{i} i \cdot n^{n-i-1}$ rooted spanning forests in K_n with i components. Taking $i = 1$ implies Cayley's theorem.

Def. Let G be a loopless graph. Then the **Laplacian** of G, $L(G)$ is a $n \times n$ symmetric matrix where:

- (i,i) is the degree of v_i
- (i,j) is the number of edges from v_i to v_j

Theorem 4.10. Kirchoff Matrix Tree Theorem: Let G be a loopless graph and $M = L(G)$ be its Laplacian. Then the number of spanning trees of G is the determinant of $M_{i,i}$ where $M_{i,i}$ is obtained from M by deleting row i and column i

5 Euler's Theorem and Hamiltonian Cycles

Lemma 5.1. Let G be a graph with $E(G) \neq \emptyset$. If G has no leaves, then it contains a cycle.

Lemma 5.2. Let G be a graph with all vertices having even degree $\implies \exists$ cycles: $C_1, C_2 ... C_k \in G$ such that each edge of G belongs to exactly one of them.

Theorem 5.3. Euler's Theorem: Let G be a connected graph with all vertices having even degree. Then, there exists a closed walk in G using each edge exactly once (a **Eularian cycle**).

Def. A **Hamiltonian cycle** is a cycle $C \subseteq G$ with $V(C) = V(G)$. There is no good way to certify that a graph has no Hamiltonian cycle.

Def. Complete bipartite graph ($K_{m,n}$): A simple graph whose vertices can be partitioned into (A,B) with $|A| = m$ and $|B| = n$. Every vertex in A is adjacent to every vertex of B and there are no other edges. $|E(K_{m,n})| = m \cdot n$.

Remark. $K_{m,n}$ has a Hamiltonian cycle iff $m = n \geq 2$. If they are not equal, there can be no cycle because vertices of A, B must alternate.

Lemma 5.4. Let G be a non-null graph with some non-empty subset X with graph $G \setminus X$ having more than $|X|$ components. Then, G has no Hamiltonian cycle.

Theorem 5.5. Dirac-Posa: Let G be a simple graph with $|V(G)| = n \geq 3$. If $deg(u) + deg(v) \geq n$ for every pair of non-adjacent vertices $u, v \in V(G)$. Then, G has a Hamiltonian cycle.

Corollary 5.6. Let G be a simple graph with $|V(G)| = n \geq 3$. If any of the following hold, then G has a Hamiltonian cycle.

a) $|E(G)| \geq \binom{n}{2} - n - 3$

b) $\forall v \in G : deg(v) \geq \frac{n}{2}$

6 Bipartite graphs

Let G be a graph, a partition (A,B) is a **bipartition** of G if every edge of G has exactly one end in A and another in B. A graph is deemed **bipartite** if it contains a bipartition.

Paths and even cycles are examples of bipartite graphs.

Theorem 6.1. Every tree is bipartite.

Theorem 6.2. For graphs G, the following are equivalent:

1. G is bipartite.
2. G contains no closed walk with an odd number of edges.
3. G contains no odd cycle.

We say H is an **induced subgraph** of G if for every $e \in E(G)$ with ends in $V(H)$, $e \in E(G)$. Equivalently, H can be obtained by deleting vertices.

Theorem 6.3. Let G be a simple graph. It is bipartite iff it contains no induced odd cycle (no induced subgraph is an odd cycle).

7 Matching in bipartite graphs

A set $M \subseteq E(G)$ is a **matching** if no edge of M is a loop and every vertex of G is incident to at most one edge in M.

We denote the **matching number** of G by $\nu(G)$, i.e. the number of edges in the matching with the most edges in G.

A set $V \subseteq V(G)$ is a **vertex cover** if each edge in $E(G)$ has an end in V.

We denote by $\tau(G)$ the minimum size of a vertex cover in G.

Lemma 7.1. For every graph G, we have that $\nu(G) \leq \tau(G)$.

For cycles, we have that:

$\nu(G) = \lfloor \frac{n}{2} \rfloor$

$\tau(G) = \lceil \frac{n}{2} \rceil$

And for complete graphs, we have that:

$\nu(G) = \lfloor \frac{n}{2} \rfloor$

$\tau(G) = n - 1$

Remark. For any simple graph, we have that $\tau(G) \geq \nu(G) \geq \frac{\tau(G)}{2}$.

Let M be a matching in G. We say that a path P is **M-alternating** if edges of P alternate between edges of M and $E(G) \setminus M$.

A path P is **M-augmenting** if $|E(P)| \geq 1$, P is M-alternating and ends of P are not incident to edges of M.

Lemma 7.2. A matching in G has maximum size iff there is no M-augmenting path.

Theorem 7.3. Konig's Theorem: For bipartite graphs, $\tau(G) = \nu(G)$

We say that $Y \subseteq V(G)$ is **covered** by a matching M if every vertex is incident to an edge of M. It is a **perfect matching** if M covers $V(G)$.

Corollary 7.4. Let G be a bipartite graph and d be a positive integer. If $\forall v \in V(G) : deg(v) = d$, then G has a perfect matching.

Theorem 7.5. Hall's Theorem: Let G have bipartition (A, B), then there is a matching in G that covers A iff $|N_G(S)| \geq |S|$ for every $S \subseteq A$.

$N_G(S)$ denotes the set of all vertices who have a neighbor in S.

8 Menger's Theorem and Separations

Def. A **separation** of a graph G is a pair (A, B) with $A \cup B = V(G)$ and there is no edge in G with one end in $A - B$ and the other in $B - A$.

To go from A to B, a path must pass through $A \cap B$.

The **order** of a separation (A, B) is $|A \cap B|$.

Remark. If (A, B) is a separation of G and P a path from $a \in A$ to $b \in B$, P contains a vertex in $A \cap B$.

Thus, we have that the max number of paths from $Q \subseteq A$ to $R \subseteq B$ is the order of the separation.

Theorem 8.1. Let G be a graph and $Q, R \subseteq V(G), k \in \mathbb{N}$. Then exactly one holds:

1. There exist paths $P_1 ... P_k$ from Q to R, pairwise vertex disjoint.
2. There exists a separation (A, B) of order $< k$ with $Q \subseteq A$ and $R \subseteq B$.

The theorem can be used to show Konig's theorem.

Theorem 8.2. Let G be a bipartite graph. If G contains no matching of size k, then G contains a vertex cover of size less than k.

Now, let's consider the case where the paths can have the same ends.

Theorem 8.3. Menger's Theorem. Let G be a graph, $s, t \in V(G)$ distinct and non-adjacent. Let k be a positive integer. Then exactly one of the following holds:

1. There exists $P_1, P_2, ... P_k$ paths in G from s to t pairwise vertex disjoint excepts for the ends ($V(P_i) \cap V(P_j) = \{s, t\}$).
2. There exists a separation (A, B) of G of order $< k$ such that $s \in A \setminus B, t \in B \setminus A$.

Def. We say that G is **k-connected** if $|V(G)| \geq k + 1$ (here or else complete graphs are infinitely connected) and $G \setminus X$ is connected for any $X \subseteq V(G)$, $|X| < k$. This means that if we remove k vertices, the graph will still be connected.

- 1-connected $\iff$ connected and non-null
  - Every tree is 1-connected but no tree is 2-connected (can always remove neighbor of leaf to disconnect).
  - Every cycle with more than 2 vertices is 2-connected.

Theorem 8.4. Let G be a k-connected graph. Then, for every pair of distinct vertices $s, t \in V(G)$, there exists paths $P_1, P_2 ... P_k$ in G such that the paths are pairwise vertex disjoint except for s, t and pairwise edge disjoint (can use a edge from s to t but only once).

Consider the case where edges break/can be deleted. In a graph G let $\delta(X)$ denote the set of all edges in G with one end in X and the other in $V(G) - X$

Def. A **line graph L(G)** of a graph G is a graph with $V(L(G)) = E(G)$ and 2 vertices of $L(G)$ are adjacent iff the corresponding edges in G share an end.

Theorem 8.5. Let G be a graph, $s, t \in V(G)$ distinct and non-adjacent. Let k be a positive integer. Then exactly one of the following holds:

- There exist k paths from s to t in G that are pairwise edge disjoint.
- There exists $X \subseteq V(G)$ such that $s \in X, t \in V(G) \setminus X$ and $|\delta(X)| < k$

9 Directed graphs and network flows

A **directed graph** (or **digraph**) G is a graph where every edge is prescribed a direction, that is for every edge e, one of its ends is called its **tail** and another its **head**. Then e is said to be directed from the tail to head.

A **directed path** P from s to t in a digraph G is a path from s to t such that following P, we get that each edge is traversed from its tail to its head.

For a digraph G and $X \subseteq V(G)$, let $\delta^+(X)$ be the set of all edges of G with tail in X and head in $V(G) \setminus X$ (i.e. that go to $V(G) \setminus X$).

Conversely, $\delta^-(X)$ is the set of all edges of G with head in X and tail in $V(G)$ (i.e. that go to X).

Lemma 9.1. Let G be a digraph, $s, t \in V(G)$. Then, exactly one of the following holds.

1. There is a directed path in G from s to t
2. There is $X \subseteq V(G)$ such that $s \in X, t \in V(G) \setminus X$ and $\delta^+(X) = \emptyset$

Let G be a digraph such that $s, t \in V(G), s \neq t$. An **s-t flow** in G is a function $\phi : E(G) \to \mathbb{R}_+$ such that for every $v \in V(G) - \{s, t\}$ we have that

$$\sum_{e \in \delta^-(v)} \phi(e) = \sum_{e \in \delta^+(v)} \phi(e)$$

i.e. that the flow into a vertex is equal to the flow out of the vertex.

The **value** of an s-t flow ϕ is $\sum_{e \in \delta^+(s)} \Phi(e) - \sum_{e \in \delta^-(s)} \Phi(e)$. The second part is to make sure we don't double count flow going back to s.

Lemma 9.2. Let ϕ be an s-t-flow on a digraph G with value k. Then, for every $X \subseteq V(G)$ such that $s \in X, t \in V(G) - X$:

$$\sum_{e \in \delta^+(X)} \phi(e) - \sum_{e \in \delta^-(X)} \phi(e) = k$$

Lemma 9.3. Let ϕ be an integral s-t flow (only positive integer flows for the edges) on a digraph G with value $k \geq 1$. Then there exists directed path $P_1, P_2 ... P_k$ in G from s to t such that every edge e belongs to at most $\phi(e)$ of these paths.

Let G be a digraph and $c : E(G) \to \mathbb{N}$ be a **capacity function** that prescribes to each edge its constraint. We say for distinct $s, t \in V(G)$ that an s-t-flow ϕ is **c-admissible** if $\phi(e) \leq c(e)$, for every $e \in E(G)$.

The question we want to answer is what is the maximum value of a c-admissible s-t-flow.

We say a path P is ϕ-augmenting if P is a path in G from s to $v \in V(G)$ (doesn't need to be directed) and:

- $\phi(e) \leq c(e) - 1$ for every $e \in E(P)$ which is used in the forward direction as we traverse P from s to v (i.e. it is used correctly).
- $\phi(e) \geq 1$ for every $e \in E(P)$ used in the other direction.

Lemma 9.4. Let G be a digraph, $s, t \in V(G)$ distinct, $c : E(G) \to \mathbb{N}$ a capacity function and ϕ be an integral, c-admissible s-t-flow of value k.

If there exists a ϕ-augmenting path in G from s to t, then there is an integral c-admissible s-t-flow on G of value $k+1$.

Theorem 9.5. Ford-Fulkerson (Max Flow- Min Cut): Let G, s, t, c be as defined above. Let $k \geq 1$ be an integer, then exactly one of the following holds:

1. There exists a c-admissible s-t-flow ϕ of value k
2. $\exists X \subseteq V(G), s \in X, t \in V(G) - X$ and $\sum_{e \in \delta^+(X)} c(e) < k$

10 Independent Sets and Ramsey Theorem

$S \subseteq V(G)$ is a **independent set** if no edge of G has both ends in S (vertices of S are not incident to loops).

We denote by $\alpha(G)$ the maximum size of an independent set in G (also known as the **independence number**).

$F \subseteq E(G)$ is an **edge cover** if every vertex of G is incident to an edge of F.

We denote by $\rho(G)$ the minimum size of an edge cover in G (only well defined if every vertex of G is incident to an edge).

G	$\nu(G)$	$\tau(G)$	$\alpha(G)$	$\rho(G)$
K_n	$\lfloor \frac{n}{2} \rfloor$	n-1	1	$\lceil \frac{n}{2} \rceil$
C_N	$\lfloor \frac{n}{2} \rfloor$	$\lceil \frac{n}{2} \rceil$	$\lfloor \frac{n}{2} \rfloor$	$\lceil \frac{n}{2} \rceil$

- $\rho(G) \geq \frac{|V(G)|}{2}$
- $\alpha(G) \leq \rho(G)$

Lemma 10.1. For any graph G, we have $\alpha(G) + \tau(G) = |V(G)|$

Theorem 10.2. Gallai equations: Let G be a connected, simple graph with $|V(G)| \geq 2$. Then:

$$\nu(G) + \rho(G) = |V(G)|$$

Corollary 10.3. Let G be a connected, bipartite, simple graph with $|V(G)| \geq 2$, then $\alpha(G) = \rho(G)$.

Let G be a simple graph. We say that $X \subseteq V(G)$ is a **clique** in G if every pair of vertices in X are adjacent.

Denote by $\omega(G)$ the size of the maximum clique in G.

Givent that $s, t = 1$, let $R(s, t)$ be the minimum positive integer n such that every simple graph G with $|V(G)| = n$ contains an independent set of size s or a clique of size t (satisfies $\alpha(G) \geq s$ or $\omega(G) \geq t$).

Theorem 10.4. Ramsey, Erdos, Szeckeres: $R(s, t)$ exists for all s and t For all $s, t \geq 2$.

$$R(s,t) \leq R(s-1,t) + R(s,t-1)$$

$$R(1,t) = 1 = R(s,1))$$

$$R(2,t) = t$$

$$R(3,3) = 6$$

Corollary 10.5. For all $s, t \geq 1 : R(s, t) \leq \binom{s+t-2}{s_1}$.

We have that:

$$(\sqrt{2})^s \leq R(s,s) \leq \binom{s+t-2}{s-1} \frac{4^s}{\sqrt{s}}$$

$$R(s,t) = R(t,s)$$

The definition of Ramsey's number is equivalent to the minimum m such that in every coloring of edges of K_n using colors red and blue, there are either s vertices pairwise joined by red edges, or t vertices pairwise joined by blue edges.

Define $R_k(s_1, s_2 ... s_k)$ as a **multicolor Ramsey number** to be the minimum n such that every coloring of edges of K_n using colors $\{1, 2 ... k\}$ there exists $1 \leq i \leq k$ and a set of s_i vertices pairwise joined by edges of color i.

Theorem 10.6. For all positive integers $k, s_1, s_2 ... s_k$, $R_k(s_1, s_2 ... s_k)$ exists (i.e. is finite).

Theorem 10.7. For every integer $k \geq 1$, there exists n such that for every coloring of $\{1...n\}$ using k colors, there exists a monochromatic solution to $x + y = z$ (i.e. $x, y, z \in \{1...n\}, x + y = z$ and x, y, z are all colored in the same color).

Example 10.1. For $k = 2, n = 5$ suffices. WLOG, assume 1 is red. If 2 is red, we have a solution (1+1 = 2), so blue. If 4 is blue, we have a solution (2+2 = 4) so red. If 5 is red (4+1=5), have a solution so blue. And now regardless of color assigned to 3, we have a solution.

Example 10.2. x+2y = z+1 does not necessarily have a monochromatic solution with 2 colors (color even numbers in one color and odd numbers in another).

Theorem 10.8. Fermat's Last Theorem: $x^n + y^n = z^n, n > 2$ has no positive integer solutions.

Fact: $x \equiv y(mod p) \iff x - y$ divisible by p.

Theorem 10.9. For every integer $m \geq 1$ there exists p_0 such that for every prime $p \geq p_0$ there exists positive integers x, y, z not divisible by p such that:

$$x^m + y^m \equiv z^m (mod p)$$

Theorem 10.10. $R(s, s) \geq 2^{\frac{s}{2}}$ for every integer $s \geq 2$.

11 Matchings in general graphs and Tutte's theorem

When does a graph have a perfect matching? A matching is perfect iff $|M| = \frac{|V(G)|}{2}$.

If $|V(G)|$ is odd, then G has no perfect matching. If a component of G has an odd number of vertices, then G also has no perfect matching.

If G is bipartite, then G has no perfect matching iff there exists a vertex cover X of G such that $|X| < \frac{|V(G)|}{2}$.

For example, consider 3 components K_4 that are connected by one vertex. Such a graph has no perfect matching.

Theorem 11.1. Tutte's Theorem: A graph G has a perfect matching iff $comp_o(G - X) \leq |X|$ for every $X \subseteq V(G)$.

Theorem 11.2. Tutte-Berge: A graph G has matching of size k iff * $comp_o(G - X) \leq |X| + |V(G)| - 2k$ for every $X \subseteq V(G)$.

11.1 is 11.2 with $k = \frac{|V(G)|}{2}$.

Def. We say that a graph G is **d-regular** if $deg(v) = d$ for every $v \in V(G)$.

- 1-regular simple graphs are matchings
- 2-regular graphs are unions of simple cycles

Def. We say $e \in E(G)$ is a **cut-edge** if $comp(G - e) = comp(G) + 1 \iff e$ does not belong to a cycle in G.

Theorem 11.3. Let G be a 3-regular graph. If G has no cut-edge, then G has a perfect matching.

12 Vertex coloring

Let G be a loopless graph. A map $\phi : V(G) \to S$ is a **k-coloring** of G if $|S| = k$ and $\phi(u) \neq \phi(v)$ for every pair u, v of adjacent vertices of G.

- Elements of S are called **colors**
- The sets of vertices which are assigned a given color are **color classes**
- Color classes are independent sets

The **chromatic number** of G, denoted by $\chi(G)$, is the minimum positive integer such that there is a k-coloring of G, i.e. G is k-colorable.

- $\chi(G) \leq 1$: Edgeless
- $\chi(G) \leq 2$: Bipartite $\iff$ no odd-cycles as subgraphs.
- $\chi(G) \leq 3$: Under the "Unique Games" hypothesis, every algorithm which does the following must sometimes take exponential time in the size of the input.
  Takes in G, and either outputs $\chi(G) \geq 4$ or $\chi(G) \leq 100000$

Lemma 12.1. Let G be a loopless graph. Then:

1. $\chi(G) \geq w(G)$ where $w(G)$ is the size of the maximum complete subgraph of G
2. $\chi(G) \geq \lceil \frac{|V(G)|}{\alpha(G)} \rceil$

Example 12.1. Applying 2. to odd cycles yields:

$$\alpha(C_{2k+1}) = k \implies \chi(C_{2k+1}) \geq \lceil \frac{2k+1}{k} \rceil = 3$$

And equality holds for every $k \geq 1$.

Let $\Delta(G)$ denote the maximum degree of a vertex in G. Let G be a loopless graph.

- $\Delta(G) = 0 \iff \chi(G) \leq 1$
- $\Delta(G) = 1 \implies \chi(G) \leq 2$
- $\Delta(G) = 2 \implies \chi(G) \leq 3$

$\chi(G) \leq \Delta(G) + 1$ for every loopless G.

A graph G is k-degenerate if every non-null subgraph H of G contains a vertex of degree at most k. Every graph G is $\Delta(G)$-degenerate.

- G is 1-degenerate $\iff$ G is a forest

Def. Greedy coloring algorithm: Algorithm for coloring graph that performs relatively well (optimal for complete graphs)

Input: Loopless graph G and an ordering $(v_1 ... v_n)$ of $V(G)$

Algorithm: Color vertices in order using integers as colors. If $v_1 ... v_i$ are colored, assign to v_{i+1} the smallest integer color which is not used by already colored neighbors of v_i

Output: A k-coloring of G for some integer k.

There is always an ordering of the vertices for which the algorithm outputs an optimal coloring.

Theorem 12.2. Let G be a loopless, k-degenerate graph for some integer $k \geq 0$. Then:

$$\chi(G) \leq k+1$$

We would like $\chi(G) \leq \Delta(G)$. However, K_n is a counter example $\chi(K_n) = n$ and $\Delta(K_n) = n-1$ as well as odd cycles that have $\chi(C_{2k+1}) = 3, \Delta(C_{2k+1}) = 2$. However, these are the only connected counter examples.

Theorem 12.3. Let G be a loopless, connected graph. If G is not a complete graph or an odd cycle, then:

$$\chi(G) \leq \Delta(G)$$

Theorem 12.4. Blaise Pascal: For every $k \geq 0$, there exists a simple graph G_k such that:

- $w(G_k) \leq 2$
- G_k is k-degenerate
- $\chi(G_k) = k+1$

Conjecture: If G is a loopless graph, then:

$$\chi(G) \leq \left\lceil \frac{w(G) + \Delta(G) + 1}{2} \right\rceil$$

This bound would be tight. Let C_5^k be 5 graphs K_k joined in a cycle. Then, we have that $w(G) = 2k$, $\Delta(G) = 3k-1$. By the conjecture and the bound from last time, we have that:

$$\lceil \frac{5k}{2} \rceil = \lceil \frac{|V(G)|}{\alpha(G)} \rceil \leq \chi(G) \leq \lceil \frac{5k}{2} \rceil$$

13 Edge Coloring

A **k-edge-coloring** of a loopless graph G is a map $\phi : E(G) \to S$ with $|S| = k$ such that $\phi(e) \neq \phi(f)$ for every pair of distinct $e, f \in E(G)$ sharing an end.

G is **k-edge-colorable** if it admits a k-edge-coloring. $\chi'(g)$ (the **edge chromatic number**) is the minimum k such that G is k-edge-colorable.

k-edge coloring of $G \iff$ k-coloring of L(G), the line graph. In particular, $\chi'(G) = \chi(L(G))$.

Proposition 13.1. For a loopless graph G, we have $\Delta(G) \leq \chi'(G) \leq 2\Delta(G) - 1$.

Lemma 13.2. Let G be a graph with $\Delta(G) \leq k$. Then G is a subgraph of some k-regular graph H. Moreover, if G is loopless (respectively bipartite) then H can also be chosen to be loopless (respectively bipartite).

Theorem 13.3. Konig: For every bipartite graph G, $\chi'(G) = \Delta(G)$.

Examples where equality doesn't hold: $\chi'(C_3) = 3$ yet $\Delta(G) = 2$. Similarly, $\chi'(C_{2k=1}) = 3$ yet $\Delta(G) = 2$.

Theorem 13.4. Vizing: For every simple graph G, $\chi'(G) \leq \Delta(G) + 1$.

For graphs without loops but where parallel edges are allowed we need a bigger bound. For example, take C_3^k to be the union of 3 cycles with k edges. Then, $\Delta(C_3^k) = 2k$ while $\chi'(C_3^k) = 3k$.

We say that $F \subseteq E(G)$ is a **2-factor** in a graph G if every vertex of G is incident to exactly two edges of F.

Theorem 13.5. Let G be a 2k-regular loopless graph then $E(G)$ can be partitioned into k 2-factors.

Theorem 13.6. Let G be a loopless graph, then $\chi'(G) \leq 3\lceil \frac{\Delta(G)}{2} \rceil$

14 Minors and Hadwiger's conjecture

H is a subgraph of G if H can be obtained from G by repeatedly vertices and/or edges.

Let e be a non-loop edge of a graph G with ends u, v. **Contracting e** corresponds to deleting e and **identifying** u, v (in the new graph, every edge incident to u or v is incident to the new vertex).

H is a **minor** of G if H can be obtained from G by either deleting vertices/edges and/or contracting edges (possibly no operations, i.e. G is a minor of G).

Equivalently, H is a minor of G if H can be obtained from a subgraph of G by contracting edges.

Proposition 14.1. A graph G contains a cycle C_k as a minor iff G contains a cycle of length at least k as a subgraph.

Consequently:

- No C_1 minor $\iff$ G is a forest
- No C_2 minor $\iff$ G is a forest with some loops added.
- No $C_3 = K_3$ minor $\iff$ G is a forest with some loops and or parallel edges added (i.e. not simple)

Hadwiger's conjecture: Let G be a loopless graph, $t \geq 1$ integer. If G has no K_t minor, then $\chi(G) \leq t-1$. Since K_{t-1} has no K_t minor and $\chi(K_{t-1}) = t-1$, such a bound would be tight.

- $t = 1$: Then you have no vertices and thus 0-colorable.
- $t = 2$: No K_2 subgraph, no loops $\implies$ no edges $\implies$ 1-colorable.
- $t = 3$: Forest, which is 2-colorable
- $t = 4$: Proved by Hadwiger
- $t = 5$: Equivalent to 4 color theorem (planar graphs don't have K_5 as minor)
- $t = 6$: Proved by Robertson, Seymour and Thomas

A **subdivision** of a graph H is a graph G obtained from H by replacing edges of H by internally vertex disjoint paths with the same ends (as the edges that paths replace).

If G contains a subdivision of H as a subgraph, then G contains H as a minor. However, the converse is not true in general (see picture).

Lemma 14.2. Let G be a 3-connected graph, then G contains a K_4 minor (and thus G contains a subdivision of K_4 as a subgraph)

Lemma 14.3. Let G be a simple, non-null graph with no K_4 minor. Then, for every clique $X \subseteq V(G), X \neq V(G), |X| \leq 2$, there exists $v \in V(G) - X$ such that $deg_G(v) \leq 2$.

Theorem 14.4. If G is a loopless graph, G has no K_4 minor, then $\chi(G) \leq 3$.

15 Planar graphs

A **drawing of a graph** G **in the plane** is a representation of G where vertices of G are represented by distinct points in the plane and edges are represented by curves joining the points corresponding to their ends, such that these curves don't intersect themselves or each other.

Such a drawing of G divides the plane into **regions** where 2 points belong to the same region if they can be joined by a curve disjoint from the drawing. There is always one unbounded region.

A graph is **planar** if it admits a planar drawing. Let $Reg(G)$ denote the number of regions in a planar drawing of G

Jordan Curve theorem: Any closed non-self intersection curve separates the plane into 2 regions. This implies that for any edge e of a planar drawing of a graph, if e belongs to a cycle, then the 2 regions on different sides of e are distinct.

Theorem 15.1. Euler's formula: Let G be a graph drawn in the plane. Then:

$$|V(G)| - |E(G)| + Reg(G) = 1 + comp(G)$$

More specifically, if G is connected, then $|V(G)| - |E(G)| + Reg(G) = 2$. If G is a forest, $Reg(G) = 1$.

The **length** of a region is the number of edges of G that belong to its boundary, with edges with both sides belonging to the boundary being counted twice.

Lemma 15.2. Let G be a simple connected graph drawn on the plane with $|V(G)| \geq 3$, then every region of G has length ≥ 3

Lemma 15.3. Let G be a simple planar graph with $|V(G)| \geq 3$. Then, $|E(G)| \leq 3|V(G)| - 6$. If G contains no K_3 subgraphs, then $|E(G)| \leq 2|V(G)| - 4$.

Corollary 15.4. The graphs K_5 and $K_{3,3}$ are not planar (count vertices/edges and see contradiction with formula).

Corollary 15.5. Let G be a simple planar graph with $|V(G)| \geq 3$, then:

$$\sum_{v \in V(G)} 6 - deg(v) \geq 12$$

Theorem 15.6. **Four Color Theorem:** If G' is simple and planar, $\chi(G) \leq 4$

Corollary 15.7. If G is simple and planar, then $\chi(G) \leq 6$.

16 Kuratowski's Theorem

Theorem 16.1. A graph G is planar iff G contains neither K_5 nor $K_{3,3}$ as a minor.

Theorem 16.2. A graph G is planar iff G contains neither a subdivision of K_5 nor a subdivision of $K_{3,3}$ as a subgraph.

To show a graph is planar, either find a drawing for it or show there is a subdivision of K_5 or $K_{3,3}$.

Theorem 16.3. **Archdeacon:** There is an explicity collection of 35 minor-minal graphs which can not be drawn in the projective plane.

Theorem 16.4. **Robertson, Seymour:** Let F be a class of graphs closed under taking minors (e.g. the class of all graphs that can be drawn on a given surface). If F is not the class of all graphs, then there exists a finite collection $H_1, H_2..H_k$ of graphs such that $G \in F$ iff G does not contain any H_i as a minor.

17 Coloring planar graphs

Theorem 17.1. **Heawood:** Let G be a loopless planar graph, then $\chi(G) \leq 5$

A graph drawn in the plane is a **triangulation** if every region is bounded by a cycle of length 3.

Let G, G^* be drawn in the plane. We say that G^* is a **planar dual** of G if

- every region of G contains exactly one vertex of G^*.
- every edge of G is crossed by exactly one edge of G^* and the drawings are otherwise disjoint.
- $|E(G)| = |E(G^*)|$.

Theorem 17.2. **Tait:** Let G be a planar triangulation and let G^* be its dual. Then, $\chi(G) \leq 4$ iff $\chi(G^*) \leq 3$.